Unspecified Journal
Volume 00, Number 0, Pages 000–000
S ????-????(XX)0000-0

ASYMPTOTIC BEHAVIOR OF IWASAWA AND CHOLESKY ITERATIONS

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Abstract. We extend, in the context of a connected real semisimple Lie group, some results on the QR iteration and the Cholesky iteration of a nonsingular matrix. A group theoretic understanding of the abstract mechanisms of the iterations is obtained.

1. Introduction

The QR iteration [2, 3, 10, 18] provides one of the most efficient methods for computing the eigenvalues of a nonsingular matrix $X \in \mathrm{GL}_n(\mathbb{C})$ with distinct eigenvalue moduli [15, p. 173-180]. The QR iteration of X is the sequence $\{X_i\}_{i\in\mathbb{N}}$ defined as follows:

$$\begin{aligned} X_1 &:= X, \\ X_i &:= R_{i-1}Q_{i-1}, \qquad i = 2, 3\ldots \end{aligned}$$

where $X_i = Q_iR_i$ denotes the QR decomposition of the matrix X_i. Since $X_i = Q_{i-1}^{-1}X_{i-1}Q_{i-1}$, the eigenvalues of each X_i are identical with those of X. It is known [6, Theorem 2.1, Theorem 5.1], [18] that if the eigenvalue moduli of X are distinct, then the sequence obtained by taking the lower triangular parts of the matrices X_i ($i \in \mathbb{N}$) converges to a diagonal matrix with diagonal entries the eigenvalues of X. When X is real, its complex eigenvalues occur in complex conjugate pairs and thus the distinct moduli assumption is often not satisfied. If the eigenvalue moduli of X are distinct except for complex conjugate pairs, then under some mild conditions [6, Theorem 2.1] the sequence $\{X_i\}$ continues to possess a certain convergence behavior, and this is valid if X is in either of the connected real semisimple Lie groups $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$ as well.

We extend these statements about the matrix X to an element of a connected real semisimple Lie group (see Theorems 4.5 and 4.7). Such a group G has Iwasawa decomposition $G = KAN$, so each $g \in G$ can

2000 Mathematics Subject Classification. Primary 15A42, 22E46 be written as $g = \mathrm{k}(g)\mathrm{a}(g)\mathrm{n}(g)$. The Iwasawa iteration $\{g_i\}_{i\in\mathbb{N}}$ of $g \in G$ is defined as follows:

$$\begin{aligned} g_1 &:= g, \\ g_i &:= \mathrm{a}(g_{i-1})\,\mathrm{n}(g_{i-1})\,\mathrm{k}(g_{i-1}), \quad i = 2, 3, \ldots. \end{aligned}$$

For $G = \mathrm{SL}_n(\mathbb{C})$, the Iwasawa decomposition (resp., iteration) is just the QR decomposition (resp., iteration) if AN is chosen to be the group of upper triangular matrices with positive diagonal entries. Our convergence results yield the QR iteration convergence statement in this special case.

Another matrix iteration is the Cholesky iteration. The Cholesky decomposition theorem asserts that any positive definite matrix Y can be written $Y = R^*R$ with R a uniquely determined upper triangular matrix with positive diagonal entries. The Cholesky iteration of a positive definite matrix Y is the sequence $\{Y_i\}_{i\in\mathbb{N}}$ of positive definite matrices defined as follows:

$$\begin{aligned} Y_1 &:= Y, \\ Y_i &:= R_{i-1}R_{i-1}^*, \quad i = 2, 3, \ldots, \end{aligned}$$

where $Y_i = R_i^*R_i$ denotes the Cholesky decomposition of the matrix Y_i. Since $Y_i = R_{i-1}Y_{i-1}R_{i-1}^{-1}$, the eigenvalues of Y_i are identical with those of Y, counting multiplicities. It is known [18] that the Cholesky iteration of Y converges to a diagonal matrix having diagonal entries the eigenvalues of Y. We give a Lie group theoretic generalization of the Cholesky iteration and obtain a corresponding convergence result (see Theorem 5.2). See [1, Theorem 11.2] and [12] for related results.

The proofs in the literature of the convergence statements for the QR iteration and the Cholesky iteration are not purely group theoretic since they usually make use of the embedding of $\mathrm{GL}_n(\mathbb{C})$ in $\mathbb{C}_{n\times n}$ to add matrices at some point (see [18] for example). The proofs of the generalizations of the convergence theorems obtained in this paper involve purely group theoretic arguments.

2. Four decompositions of G

From now on (unless we say otherwise) G denotes a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. In this section, we describe four well-known decompositions of G (or an element of G), namely the Cartan decomposition, the Iwasawa decomposition, the Bruhat decomposition, and the complete multiplicative Jordan decomposition.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}$ [8]. Let $K \subset G$ be the connected subgroup corresponding to $\mathfrak{k}$. Then K is closed and $\mathrm{Ad}\,(K)$ is a maximal compact subgroup of $\mathrm{Ad}\,(G)$ [4, p. 402]. Set $P := \exp \mathfrak{p}$. The map

$$K \times P \to G, \qquad (k,p) \mapsto kp$$

is a diffeomorphism. In particular $G = KP$ and every element $g \in G$ can be uniquely written as

$$g = kp, \qquad k \in K,\ p \in P.$$

This decomposition of G (resp., $\mathfrak{g}$) is the (global) Cartan decomposition. The map $\Theta : G \to G$

$$\Theta(kp) = kp^{-1}, \quad k \in K,\ p \in P,$$

is an automorphism of G [8, p. 387]. The map $* : G \to G$ defined by

$$g^* := \Theta(g^{-1}) = pk^{-1}, \quad g \in G,$$

is a diffeomorphism. When $G = \mathrm{SL}_n(\mathbb{C})$, X^* is simply the complex conjugate transpose of $X \in \mathrm{SL}_n(\mathbb{C})$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. Then $A := \exp \mathfrak{a}$ is the analytic subgroup of G corresponding to $\mathfrak{a}$. The Weyl group W of $(\mathfrak{a}, \mathfrak{g})$ is defined as $W := M'/M$, where $M' \subset K$ is the normalizer of A in K, and $M \subset K$ is the centralizer of A in K. This group operates naturally on $\mathfrak{a}$ and A and the map $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

Let $\mathfrak{a}_+$ be a (closed) Weyl chamber in $\mathfrak{a}$ and set $A_+ := \exp \mathfrak{a}_+$. The set $\mathfrak{a}_+$ corresponds to a choice of positive roots in the (restricted) root system of $\mathfrak{g}$. Let $\mathfrak{n}$ be the sum of all positive root spaces of $\mathfrak{g}$ and set $N := \exp \mathfrak{n}$. Then G has the Iwasawa decomposition $G = KAN$ [8]. For $g \in G$, we have

$$g = \mathrm{k}(g)\, \mathrm{a}(g)\, \mathrm{n}(g)$$

where $\mathrm{k}(g) \in K$, $\mathrm{a}(g) \in A$, $\mathrm{n}(g) \in N$ are uniquely determined by g. When $G = \mathrm{SL}_n(\mathbb{C})$, the Iwasawa decomposition of $X \in G$ gives the QR decomposition of X if we choose AN as the group of upper triangular matrices with positive diagonal elements and put $Q = \mathrm{k}(g)$, $R = \mathrm{a}(g)\mathrm{n}(g)$. Though the Iwasawa decomposition of $g \in \mathrm{SL}_n(\mathbb{R})$ coincides with that when g is viewed as element in $\mathrm{SL}_n(\mathbb{C})$, it is not necessarily true for any semisimple subgroup of G, e.g., $\mathrm{Sp}_n(\mathbb{R}), \mathrm{Sp}_n(\mathbb{C}) \subset \mathrm{SL}_{2n}(\mathbb{C})$ [16].

For each $s \in W$, we denote by $m_s \in M'$ a representative such that $s = m_s M$. Moreover, for $s = 1$, we choose the identity of G for m_s. The Bruhat decomposition of G is

$$G = \bigcup_{s \in W} N^- m_s MAN, \tag{1}$$

a disjoint union [11, p. 117]. Here, $N^- := \exp \mathfrak{n}^-$ where $\mathfrak{n}^-$ is the sum of the negative root spaces in $\mathfrak{g}$ (or, equivalently, $N^- = m_\gamma N m_\gamma^{-1}$, with γ the longest element in W). So for each $g \in G$, there exists a unique $s \in W$ such that $g \in N^- m_s MAN$. It is easy to see that MA normalizes N and N^-, M' normalizes M and A, and M centralizes A.

Another common version of the Bruhat decomposition of G is

$$G = \bigcup_{s \in W} MAN m_s MAN \tag{2}$$

[8, p. 398]. This is equivalent to the preceding one with more flexibility on the choices of the M and A components. For example, if $g \in G$ has the Bruhat decomposition in the form (1): $g = n^- m_s man \in N^- m_s MAN$, then $m_\gamma^{-1} g$ has the Bruhat decomposition in the form (2):

$$\begin{aligned} m_\gamma^{-1} g &= (m_\gamma^{-1} n^- m_\gamma)(m_\gamma^{-1} m_s) man \\ &\in N m_{\gamma^{-1}s} MAN \subset MAN m_{\gamma^{-1}s} MAN, \end{aligned}$$

and vice versa. For simplicity, the Bruhat decompositions in this article are always in the form (1).

The last decomposition we discuss is the complete multiplicative Jordan decomposition (CMJD). An element $g \in G$ is **elliptic** if $\mathrm{Ad}\,(g) \in \mathrm{Aut}\,\mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1; an element $g \in G$ is **hyperbolic** if $g = \exp X$, where $X \in \mathfrak{g}$ is real semisimple, which is to say ad $X \in \mathrm{End}\,\mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with all eigenvalues real; an element $g \in G$ is **unipotent** if $g = \exp X$, where $X \in \mathfrak{g}$ is nilpotent, which is to say all eigenvalues of ad $X \in \mathrm{End}\,\mathfrak{g}$ are zero.

Each $g \in G$ can be uniquely written as $g = ehu$, where e is elliptic, h is hyperbolic, u is unipotent, and the three elements e, h, u commute [9, Proposition 2.1]. This is the complete multiplicative Jordan decomposition of g. We write

$$g = \mathrm{e}(g)\ \mathrm{h}(g)\ \mathrm{u}(g).$$

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to an element of A_+; in this case, such an element of A_+ is uniquely determined and we denote it by $\mathrm{b}(h)$ [9, Proposition 2.4]. For $g \in G$, we define

$$\mathrm{b}(g) := \mathrm{b}(\mathrm{h}(g)) \in A_+.$$

3. Regular elements

An element $b \in A_+$ is **regular** if $\alpha(\log b) > 0$ for all positive roots α, that is, b is in the interior A_+° of A_+. When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$, the

CMJD of $g \in G$ is given in [4, p. 430-431]. Moreover, $\mathrm{b}(g)$ is regular if and only if g has distinct eigenvalue moduli, which implies that g is diagonalizable, that is, the unipotent part $\mathrm{u}(g) = 1$. Proposition 3.2 is an extension of this result in the context of a connected real semisimple Lie group G. Its proof requires a lemma.

Lemma 3.1. Let $b \in A_+$ be a regular element. Then the centralizer of b in G is $Z_G(b) = MA$.

Proof. Let $z \in Z_G(b)$ so that $b = z^{-1}bz$. Let

$$z = n^- m_s man$$

be a Bruhat decomposition of z, where $n^- \in N^-$, $s \in W$, $m \in M$, $a \in A$, $n \in N$. Then $s^{-1} \cdot b = b$ by [7, Corollary 3.6]. So $s = 1$ since $b \in A_+$ is regular and W permutes the Weyl chambers simply transitively. Thus $z = n^- man$ and $bn^- man = n^- manb$. Since A normalizes N^-,

$$\begin{aligned} N^- &\ni b^{-1}(n^-)^{-1}bn^- \\ &= b^{-1}manbn^{-1}a^{-1}m^{-1} \\ &= ma(b^{-1}nbn^{-1})a^{-1}m^{-1} \in N \end{aligned} \tag{3}$$

because MA normalizes N. Now $N^- \cap N = \{1\}$. So $bn^- = n^- b$ from (3). Write $b = \exp X$ for $X \in \mathfrak{a}_+^\circ$ (the interior of $\mathfrak{a}_+$) and $n^- = \exp T$ for $T \in \mathfrak{n}^-$. Then

$$\exp[\mathrm{Ad}\ (b)(T)] = b(\exp T)b^{-1} = \exp T.$$

Since the exponential map is a diffeomorphism [8, p. 68, p. 317] of $\mathfrak{n}^-$ onto N^-, $\mathrm{Ad}\ (b)(T) = T$. Therefore, $e^{\mathrm{ad}\ X}(T) = T$, implying $[X, T] = \mathrm{ad}\ X(T) = 0$. So $T = 0$ since $X \in \mathfrak{a}_+^\circ$, and thus $n^- = 1$. Likewise, $n = 1$. Therefore, $z = ma \in MA$. □

Proposition 3.2. Let $g \in G$ such that $\mathrm{b}(g) \in A_+$ is regular. Then the unipotent component $\mathrm{u}(g)$ in the CMJD of g is the identity and there is $y \in G$ such that $y\mathrm{h}(g)y^{-1} = \mathrm{b}(g)$ and $y\mathrm{e}(g)y^{-1} \in M$.

Proof. Let $g = ehu$ be the CMJD of g. There exists $y \in G$ such that $b := yhy^{-1} = \mathrm{b}(g)$. Since yey^{-1} and yuy^{-1} commute with b, we have $yey^{-1}, yuy^{-1} \in MA$ by Lemma 3.1. The elements of M are elliptic and the elements of A are hyperbolic. Moreover, the elements of M commute with those of A, so by the uniqueness of CMJD we have $yey^{-1} \in M$ and $yuy^{-1} = 1$. □

4. Asymptotic behavior of Iwasawa iteration

The present work was largely motivated by the following result from [6, Theorem 2.1]. Let $X \in \mathrm{GL}_n(\mathbb{R})$ be a matrix such that the eigenvalues of X have distinct moduli except for the conjugate pairs. It is known that [5, p. 152] X admits the decomposition $X = Y^{-1}DY$, with $Y \in \mathrm{GL}_n(\mathbb{R})$ and

$$D := \mathrm{diag}\,(\lambda_1 E_{\theta_1}, \cdots, \lambda_m E_{\theta_m}), \qquad \lambda_1 > \cdots > \lambda_m > 0,$$

$\theta_i \in [0, \pi]$, where

$$E_0 := 1, \qquad E_\pi := -1, \qquad E_\theta := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (0 < \theta < \pi).$$

The matrix Y has a Bruhat decomposition $Y = L\omega U$, where L is unit lower triangular, U is upper triangular, and ω is a permutation matrix uniquely determined by Y. In the statement of the following theorem, X_k denotes the kth term of the QR iteration of X.

Theorem 4.1. [6] Let $X \in \mathrm{GL}_n(\mathbb{R})$ be a matrix such that the eigenvalues of X have distinct moduli except for the conjugate pairs. Let $\gamma = (\gamma_1, \cdots, \gamma_m)$ where γ_i is the size of E_{θ_i}, $i = 1, \ldots, m$. Let $[X]_\gamma$ be the block form of X corresponding to the partition γ. If $Y = L\omega U$ and $[\omega]_\gamma$ is block diagonal (for example, if ω is the identity matrix), then the strictly lower triangular block part of $[X_k]_\gamma$ converges to zero and the eigenvalues of the i-th diagonal block of $[X_k]_\gamma$ converge to the eigenvalues of $\lambda_i E_{\theta_i}$.

This theorem holds in particular with the additional assumption that $X \in \mathrm{SL}_n(\mathbb{R})$. Theorem 4.5 below generalizes this special case of the theorem to the case where X is an element of an arbitrary connected real semisimple Lie group G.

The sequence $\{X_k\}$ itself need not converge. There are examples in [6, Section 4] that show this to be the case. The examples in [6, Section 3] nevertheless show that certain patterns arise, some of which can be explained by Theorem 4.5 below (see Remark 4.6).

The following sequence $\{g_i\}_{i\in\mathbb{N}}$ is called the Iwasawa iteration of $g \in G$:

$$\begin{aligned} g_1 &:= g, \\ g_i &:= \mathrm{a}(g_{i-1})\mathrm{n}(g_{i-1})\mathrm{k}(g_{i-1}), \quad i = 2, 3, \ldots \end{aligned}$$

Proposition 3.2 shows that $g \in G$ has regular $\mathrm{b}(g)$ if and only if g is conjugate to an element in MA_+°, which is a subset of

$$S := \{cb \mid c \in K,\ b \in A_+,\ cb = bc\}.$$

In this section, we first establish a result concerning the asymptotic behavior of $\{\mathrm{k}(g_i)\}_{i\in\mathbb{N}}$ for those $g \in G$ conjugate to an element of S (such a g has $\mathrm{u}(g) = 1$ in the CMJD of g). Then we describe the precise asymptotic behavior of $\{\mathrm{k}(g_i)\}_{i\in\mathbb{N}}$, $\{\mathrm{a}(g_i)\}_{i\in\mathbb{N}}$, and $\{\mathrm{n}(g_i)\}_{i\in\mathbb{N}}$ under the assumption that $\mathrm{b}(g)$ is regular.

Lemma 4.2. Let $g \in G$ and let $\{g_i\}_{i\in\mathbb{N}}$ be the Iwasawa iteration of g. The Iwasawa decompositions of $\{g^i\}_{i\in\mathbb{N}}$ can be described in terms of the Iwasawa decompositions of $\{g_i\}_{i\in\mathbb{N}}$ as follows:

$$\mathrm{k}(g^i) = \mathrm{k}(g_1)\mathrm{k}(g_2)\cdots\mathrm{k}(g_i), \tag{4}$$

$$\mathrm{a}(g^i)\mathrm{n}(g^i) = \mathrm{a}(g_i)\mathrm{n}(g_i)\mathrm{a}(g_{i-1})\mathrm{n}(g_{i-1})\cdots\mathrm{a}(g_1)\mathrm{n}(g_1). \tag{5}$$

Conversely, the Iwasawa decompositions of $\{g_i\}_{i\in\mathbb{N}}$ can be described in terms of the Iwasawa decompositions of $\{g^i\}_{i\in\mathbb{N}}$ as follows:

$$\mathrm{k}(g_i) = \mathrm{k}(g^{i-1})^{-1}\,\mathrm{k}(g^i), \tag{6}$$

$$\mathrm{a}(g_i) = \mathrm{a}(g^i)\,\mathrm{a}(g^{i-1})^{-1}, \tag{7}$$

$$\mathrm{n}(g_i) = \mathrm{a}(g^{i-1})\,\mathrm{n}(g^i)\,\mathrm{n}(g^{i-1})^{-1}\,\mathrm{a}(g^{i-1})^{-1}. \tag{8}$$

Moreover, g_i is conjugate to g by

$$g_i = \mathrm{k}(g^{i-1})^{-1}\,g\,\mathrm{k}(g^{i-1}) = \mathrm{a}(g^{i-1})\,\mathrm{n}(g^{i-1})\,g\,\mathrm{n}(g^{i-1})^{-1}\,\mathrm{a}(g^{i-1})^{-1}. \tag{9}$$

Proof. First, we prove (4) and (5) by induction. Each equation holds when $i = 1$. Assume that both equations hold for $i = t - 1$ and all $g \in G$. We have

$$\mathrm{k}(g_1)\mathrm{k}(g_2)\cdots\mathrm{k}(g_t) \in K, \quad \text{and} \quad \mathrm{a}(g_t)\mathrm{n}(g_t)\cdots\mathrm{a}(g_2)\mathrm{n}(g_2)\mathrm{a}(g_1)\mathrm{n}(g_1) \in AN$$

since A normalizes N. Applying the induction hypothesis to g_2, we get

$$\begin{aligned}
\mathrm{k}(g_1)\mathrm{k}(g_2)\cdots\mathrm{k}(g_t)&\mathrm{a}(g_t)\mathrm{n}(g_t)\cdots\mathrm{a}(g_2)\mathrm{n}(g_2)\mathrm{a}(g_1)\mathrm{n}(g_1)\\
&= \mathrm{k}(g_1)g_2^{t-1}\mathrm{a}(g_1)\mathrm{n}(g_1)\\
&= \mathrm{k}(g_1)\,[\mathrm{a}(g_1)\mathrm{n}(g_1)\mathrm{k}(g_1)]^{t-1}\,\mathrm{a}(g_1)\mathrm{n}(g_1)\\
&= [\mathrm{k}(g_1)\mathrm{a}(g_1)\mathrm{n}(g_1)]^t \;=\; g^t.
\end{aligned}$$

So (4) and (5) hold for $i = t$ and hence for all $i \in \mathbb{N}$.

Next, we prove (6), (7), and (8). Clearly (4) implies (6). Because A normalizes N, (5) gives

$$\mathrm{a}(g^i) = \mathrm{a}(g_i)\mathrm{a}(g_{i-1})\cdots\mathrm{a}(g_1),$$

so (7) holds. Then by (5) and (7),

$$\begin{aligned}
\mathrm{n}(g_i) &= \mathrm{a}(g_i)^{-1}\,\mathrm{a}(g^i)\mathrm{n}(g^i)\,[\mathrm{a}(g^{i-1})\mathrm{n}(g^{i-1})]^{-1}\\
&= \mathrm{a}(g^{i-1})\mathrm{n}(g^i)\mathrm{n}(g^{i-1})^{-1}\mathrm{a}(g^{i-1})^{-1},
\end{aligned}$$

so (8) holds.

Finally, by (6), (7), and (8),

$$\begin{aligned} g_i &= \mathrm{k}(g_i)\mathrm{a}(g_i)\mathrm{n}(g_i) \\ &= \mathrm{k}(g^{i-1})^{-1}\mathrm{k}(g^i)\mathrm{a}(g^i)\mathrm{n}(g^i)\mathrm{n}(g^{i-1})^{-1}\mathrm{a}(g^{i-1})^{-1} \\ &= \mathrm{k}(g^{i-1})^{-1}g^i\mathrm{n}(g^{i-1})^{-1}\mathrm{a}(g^{i-1})^{-1} \\ &= \mathrm{k}(g^{i-1})^{-1}g\mathrm{k}(g^{i-1}) \\ &= \mathrm{a}(g^{i-1})\,\mathrm{n}(g^{i-1})\,g\,\mathrm{n}(g^{i-1})^{-1}\,\mathrm{a}(g^{i-1})^{-1}, \end{aligned}$$

so (9) holds. □

Let Σ^+ denote the set of positive roots, and $\Sigma^- = -\Sigma^+$ denote the set of negative roots. Then

$$\begin{aligned} \mathfrak{a}_+ &= \{H \in \mathfrak{a} \mid \alpha(H) \geq 0 \text{ for all } \alpha \in \Sigma^+\}, \\ \mathfrak{n}^- &= \sum_{\alpha\in\Sigma^-} \mathfrak{g}_\alpha, \end{aligned}$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root space of α. For $H \in \mathfrak{a}_+$, put

$$\begin{aligned} \mathfrak{n}_H^0 &:= \sum_{\alpha\in P_H^0} \mathfrak{g}_\alpha \quad \text{where} \quad P_H^0 := \{\alpha \in \Sigma^- \mid \alpha(H) = 0\}; \\ \mathfrak{n}_H &:= \sum_{\alpha\in P_H} \mathfrak{g}_\alpha \quad \text{where} \quad P_H := \{\alpha \in \Sigma^- \mid \alpha(H) < 0\}. \end{aligned}$$

Then the algebra $\mathfrak{n}^-$ decomposes as a direct sum $\mathfrak{n}^- = \mathfrak{n}_H^0 \oplus \mathfrak{n}_H$. Denote by

$$\pi_H^0 : \mathfrak{n}^- \to \mathfrak{n}_H^0$$

the projection onto the first summand. Note that when $\exp H \in A_+$ is regular, we have $P_H^0 = \emptyset$, $\mathfrak{n}_H^0 = 0$, and π_H^0 maps $\mathfrak{n}^-$ to 0.

Lemma 4.3. Let $b \in A_+$ and $\ell \in N^-$. Denote $H := \log b \in \mathfrak{a}_+$ and $L := \log \ell \in \mathfrak{n}^-$. Then

$$\lim_{i\to\infty} b^i \ell b^{-i} = \exp \pi_H^0(L) \in N^-.$$

In particular, if b is regular, then $\lim_{i\to\infty} b^i\ell b^{-i} = 1$ ([4, p. 278]).

Proof. Fix $i > 0$. Using [8, (1.90), (1.94)] we get

$$b^i\ell b^{-i} = \exp[\mathrm{Ad}\,(b^i)(L)] = \exp[e^{\mathrm{ad}\,(iH)}(L)] = \exp L_i,$$

where $L_i := e^{\mathrm{ad}\,(iH)}(L)$. For any root α and $L_\alpha \in \mathfrak{g}_\alpha$, we have

$$\mathrm{ad}\,(iH)(L_\alpha) = [iH, L_\alpha] = i\alpha(H)L_\alpha.$$

So

$$e^{\mathrm{ad}\,(iH)}(L_\alpha) = \sum_{j=0}^{\infty} \frac{[\mathrm{ad}\,(iH)]^j}{j!}(L_\alpha) = \sum_{j=0}^{\infty} \frac{[i\alpha(H)]^j}{j!} L_\alpha = e^{i\alpha(H)} L_\alpha.$$

Since $L \in \mathfrak{n}^-$, we have $L = \sum_{\alpha\in\Sigma^-} L_\alpha$, with $L_\alpha \in \mathfrak{g}_\alpha$ for each α. So

$$L_i = e^{\mathrm{ad}\,(iH)}(L) = \sum_{\alpha\in\Sigma^-} e^{i\alpha(H)} L_\alpha.$$

Since $\alpha(H) < 0$ for all $\alpha \in \Sigma^-$ with $\alpha(H) \neq 0$, it follows that

$$\lim_{i\to\infty} b^i \ell b^{-i} = \lim_{i\to\infty} \exp L_i = \exp \lim_{i\to\infty} L_i = \exp \pi_H^0(L)$$

as desired. □

Lemma 4.4. Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in G, such that $\lim_{i\to\infty} x_i = 1$ and $\{\mathrm{Ad}\,(y_i)\}_{i\in\mathbb{N}}$ is in a compact subset of $\mathrm{Ad}\,(G)$. Then

$$\lim_{i\to\infty} y_i x_i y_i^{-1} = 1.$$

Proof. The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism. Therefore, since $\lim_{i\to\infty} x_i = 1$, there exist $N \in \mathbb{Z}^+$ and $X_i \in \mathfrak{g}$ $(i > N)$, such that $x_i = \exp X_i$ for $i > N$, and $\lim_{i\to\infty} X_i = 0$. Then $y_i x_i y_i^{-1} = \exp[\mathrm{Ad}\,(y_i)(X_i)]$ for $i > N$.

Suppose to the contrary that $\lim_{i\to\infty} y_i x_i y_i^{-1} \neq 1$. Then

$$\lim_{i\to\infty} \mathrm{Ad}\,(y_i)(X_i) \neq 0,$$

so there exist an open neighborhood U of $0 \in \mathfrak{g}$ and a subsequence $\{t_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$, such that $t_i > N$ and $\mathrm{Ad}\,(y_{t_i})(X_{t_i}) \notin U$ for all $i \in \mathbb{N}$. However, by assumption $\{\mathrm{Ad}\,(y_{t_i})\}_{i\in\mathbb{N}}$ is in a compact subset of $\mathrm{Ad}\,(G)$. So there exist a subsequence $\{s_i\}_{i\in\mathbb{N}}$ of $\{t_i\}_{i\in\mathbb{N}}$ and $y \in G$ such that $\lim_{i\to\infty} \mathrm{Ad}\,(y_{s_i}) = \mathrm{Ad}\,(y)$. This implies that $\lim_{i\to\infty} \mathrm{Ad}\,(y_{s_i})(X_{s_i}) = \mathrm{Ad}\,(y)(0) = 0$, which contradicts that $\mathrm{Ad}\,(y_{s_i})(X_{s_i}) \notin U$ for all i. Therefore, $\lim_{i\to\infty} y_i x_i y_i^{-1} = 1$. □

The following two main theorems present the convergence patterns of $\mathrm{k}(g_i)$, $\mathrm{a}(g_i)$, and $\mathrm{n}(g_i)$ for some special $g \in G$.

Theorem 4.5. Let $g \in G$ and assume that $ygy^{-1} = cb$ for some $y \in G$, $c \in K$, $b \in A_+$, and such that $cb = bc$. Suppose that y has a Bruhat decomposition

$$y = n^- m_s man \in N^- m_s MAN.$$

Let $n_0^- := \exp \pi_H^0(L)$ where $H := \log b \in \mathfrak{a}_+$ and $L := \log n^- \in \mathfrak{n}^-$. Put

$$\tilde{c}_s := (n_0^- m_s m)^{-1} c (n_0^- m_s m). \tag{10}$$

Then there exists a sequence $\{d_i\}_{i\in\mathbb{N}}$ in the set $AN\tilde{c}_sAN\cap K$ such that

$$\lim_{i\to\infty} \mathrm{k}(g_i)d_i^{-1} = 1.$$

Proof. Let

$$y^{-1}n_0^- m_s m = \bar{k}\bar{a}\bar{n} \qquad \text{and} \qquad \bar{a}\bar{n}\tilde{c}_s^i = k_i a_i n_i \tag{11}$$

be the Iwasawa decompositions of the indicated elements. For $i \in \mathbb{N}$ define

$$x_i := k_i^{-1}\bar{a}\bar{n}(n_0^- m_s m)^{-1}c^i(b^i n^- b^{-i}{n_0^-}^{-1})c^{-i}(n_0^- m_s m)\bar{n}^{-1}\bar{a}^{-1}k_i, \tag{12}$$

and let

$$x_i = \hat{k}_i\hat{a}_i\hat{n}_i$$

be the Iwasawa decomposition of x_i. Put

$$b_s := m^{-1}m_s^{-1}bm_sm = m_s^{-1}bm_s = s^{-1}\cdot b \in A. \tag{13}$$

Then the components of the Iwasawa decomposition of g^i are

$$\mathrm{k}(g^i) = \bar{k}k_i\hat{k}_i \in K, \tag{14}$$

$$\mathrm{a}(g^i) = \hat{a}_i a_i b_s^i a \in A, \tag{15}$$

$$\mathrm{n}(g^i) = a^{-1}b_s^{-i}a_i^{-1}\hat{n}_i a_i n_i b_s^i an \in N. \tag{16}$$

as can be verified by a straightforward computation to see that the product of the indicated elements is g^i.

By (6),

$$\mathrm{k}(g_i) = \hat{k}_{i-1}^{-1}k_{i-1}^{-1}k_i\hat{k}_i.$$

Now $\lim_{i\to\infty} b^i n^- b^{-i}{n_0^-}^{-1} = 1$ by Lemma 4.3, and since $\mathrm{Ad}\,(c^i)$ and $\mathrm{Ad}\,(k_i^{-1})$ are in the compact set $\mathrm{Ad}\,K$, it follows from (12) and Lemma 4.4 that $\lim_{i\to\infty} x_i = 1$. In turn,

$$\lim_{i\to\infty}\hat{k}_i = \lim_{i\to\infty}\hat{a}_i = \lim_{i\to\infty}\hat{n}_i = 1 \tag{17}$$

since the multiplication map

$$K\times A\times N \to KAN = G$$

is a homeomorphism. Therefore, if

$$d_i := k_{i-1}^{-1}k_i = a_{i-1}n_{i-1}\tilde{c}_s n_i^{-1}a_i^{-1} \in AN\tilde{c}_sAN\cap K,$$

then

$$\lim_{i\to\infty}\mathrm{k}(g_i)d_i^{-1} = \lim_{i\to\infty}\hat{k}_{i-1}^{-1}d_i\hat{k}_i d_i^{-1} = (\lim_{i\to\infty}\hat{k}_{i-1}^{-1})(\lim_{i\to\infty} d_i\hat{k}_i d_i^{-1}) = 1$$

using Lemma 4.4 again. □

Remark 4.6.

(1) The theorem can be interpreted as saying that, even though the sequence $\mathrm{k}(g_i)$ might not converge, it at least gets ever closer to a sequence restricted to the set $AN\tilde{c}_sAN \cap K$. In turn, $g_i = \mathrm{k}(g_i)\mathrm{a}(g_i)\mathrm{n}(g_i)$ gets ever closer to a sequence in the set $(AN\tilde{c}_sAN \cap K)AN \subseteq AN\tilde{c}_sAN$.

(2) We can now recover the special case of Theorem 4.1 with $G = \mathrm{SL}_n(\mathbb{R})$.We may pick $K = \mathrm{SO}(n)$ and $A_+ \subset \mathrm{SL}_n(\mathbb{R})$ the set of diagonal matrices of nonincreasing diagonal entries. Assume the hypotheses of that theorem and put $g = X$. By assumption, there exists $y \in G$ such that $ygy^{-1} = cb$ where $c = \mathrm{diag}(E_{\theta_1}, \ldots, E_{\theta_m}) \in K$ and $b = \mathrm{diag}(\lambda_1 I_{\gamma_1}, \ldots, \lambda_m I_{\gamma_m}) \in A_+$ (both block diagonals relative to the partition γ; b is not regular in general). Note that $cb = bc$. Also by assumption, we have a decomposition $y = L\omega U$, with $L \in N^-$, $\omega M \in W$, and $U = DU' \in AN$, so putting $n^- = L$, $m_s = \omega$, $m = 1$, $a = D$, and $n = U'$, we have $y = n^- m_s man \in N^- m_s MAN$ and the hypotheses of Theorem 4.5 are met. The conclusion is that $\mathrm{k}(g_i)$ gets ever closer to a sequence in the set $AN\tilde{c}_sAN$. Now n_0^- is block diagonal (relative to the partition γ), so it follows that $\tilde{c}_s$ is block diagonal as well, and so $AN\tilde{c}_sAN$ consists of upper triangular block matrices. Therefore, the lower triangular blocks of the matrices $X_i = g_i$ tend to zero in agreement with Theorem 4.1

(3) Theorem 4.5 can also be used to explain some of the patterns found in [6, Section 3]. (The matrix ω in that section is not in $\mathrm{SL}_4(\mathbb{R})$ so the example does not satisfy our requirement that G be semisimple as is, but one can arrange to have all of the indicated matrices in $\mathrm{SL}_4(\mathbb{R})$, without affecting the observed patterns, by making the replacements $Y \to I_4^- Y$, $L \to I_4^- L I_4^-$, $\omega \to I_4^- \omega$, and $c \to -c$, where $I_4^- := \mathrm{diag}(-1, 1, 1, 1)$.) Consider the situation (1) in [6, Section 3]. The argument above applies here except that m_s $(= \omega)$, and therefore $\tilde{c}_s$, is no longer block diagonal relative to $\gamma = (2, 2)$. But at least the $(4, 1)$-entry of $\tilde{c}_s$ is zero, and the same then applies to every matrix in $AN\tilde{c}_sAN$. This explains the observed pattern that the $(4, 1)$-entries in the Iwasawa iterations approach zero. This argument is valid for the cases (4) and (5) as well. Similarly, in case (2), $\gamma = (2, 1, 1)$, which implies that the $(4, 1)$-, $(4, 2)$-, and $(4, 3)$-entries of $\tilde{c}_s$ are each zero and hence these entries of the Iwasawa iterations must each approach zero. An argument similar to this handles case (3) as well.

The counterpart in the Lie group setting of [6, Theorem 5.1] is given in the following result.

Theorem 4.7. Let the notation be as in Theorem 4.5 and suppose that $g \in G$ with $\mathrm{b}(g)$ regular. Then the components of the Iwasawa decomposition of g_i have the following asymptotic behaviors:

(1) $\lim_{i\to\infty} \mathrm{k}(g_i) = c_s \in M$, where $c_s := (m_s m)^{-1} c (m_s m)$,
(2) $\lim_{i\to\infty} \mathrm{a}(g_i) = b_s \in A$, where $b_s := (m_s m)^{-1} b (m_s m) = s^{-1}\cdot b$ as in (13),
(3) $\lim_{i\to\infty} \left(c_s^i \mathrm{n}(g_i) c_s^{-i}\right) = b_s^{-1}\bar{a}\bar{n} b_s c_s \bar{n}^{-1}\bar{a}^{-1} c_s^{-1} = b_s^{-1}\bar{k}^{-1} g \bar{k} c_s^{-1} \in N$, where $\bar{k}$, $\bar{a}$ and $\bar{n}$ are as in (11).

Proof. Since $\mathrm{b}(g)$ is regular, Proposition 3.2 shows that there is $y \in G$ such that $ygy^{-1} = cb$, where $c \in M \subset K$ and $b = \mathrm{b}(g) \in A_+$. Thus the assumption on g in Theorem 4.5 is satisfied. Moreover, $n_0^- = 1$ by Lemma 4.3. So in (10), $\tilde{c}_s = (m_s m)^{-1} c (m_s m) = c_s \in M$. Since M normalizes AN and A normalizes N,

$$AN\tilde{c}_s AN \cap K = c_s AN \cap K = \{c_s\}$$

by the uniqueness of the Iwasawa decomposition. In Theorem 4.5, we have $d_i = c_s$ and so $\lim_{i\to\infty} \mathrm{k}(g_i) = c_s$.

The second decomposition in (11) shows that

$$k_i a_i n_i = c_s^i \bar{a}(c_s^{-i}\bar{n} c_s^i),$$

so $a_i = \bar{a}$ and $n_i = c_s^{-i}\bar{n}c_s^i$. By (7), (15), and (17),

$$\mathrm{a}(g_i) = \mathrm{a}(g^i)\mathrm{a}(g^{i-1})^{-1} = \hat{a}_i\hat{a}_{i-1}^{-1} a_i a_{i-1}^{-1} b_s = \hat{a}_i\hat{a}_{i-1}^{-1} b_s \;\to\; b_s \quad \text{as } \; i \to \infty.$$

By (8), (15), (16), (17), and Lemma 4.4,

$$\begin{aligned} c_s^i n(g_i) c_s^{-i} &= c_s^i a(g^{i-1}) n(g^i) n(g^{i-1})^{-1} a(g^{i-1})^{-1} c_s^{-i} \\ &= c_s^i b_s^{-1} a_{i-1} a_i^{-1} \hat{a}_{i-1}\hat{n}_i a_i n_i b_s n_{i-1}^{-1} a_{i-1}^{-1} \hat{n}_{i-1}^{-1}\hat{a}_{i-1}^{-1} c_s^{-i} \\ &\to b_s^{-1}\bar{a}\bar{n} b_s c_s \bar{n}^{-1}\bar{a}^{-1}c_s^{-1} \quad \text{as } \; i\to\infty. \end{aligned}$$

Moreover, the first decomposition in (11) and $n_0^- = 1$ imply that

$$b_s^{-1}\bar{a}\bar{n} b_s c_s \bar{n}^{-1}\bar{a}^{-1} c_s^{-1} = b_s^{-1}\bar{k}^{-1} g\bar{k} c_s^{-1}.$$

This completes the proof. □

Remark 4.8.

(1) When $\mathrm{b}(g)$ is regular, Theorem 4.7 shows that $\{\mathrm{k}(g_i)\}_{i\in\mathbb{N}}$ and $\{\mathrm{a}(g_i)\}_{i\in\mathbb{N}}$ converge. It follows that $\{\mathrm{n}(g_i)\}_{i\in\mathbb{N}}$ converges if and only if $\{g_i\}_{i\in\mathbb{N}}$ converges.

(2) When $\mathrm{b}(g)$ is not regular, $\{\mathrm{k}(g_i)\}_{i\in\mathbb{N}}$ and $\{\mathrm{a}(g_i)\}_{i\in\mathbb{N}}$ may not converge. For example, let $a > 0$ and

$$g = \begin{bmatrix} 0 & a \\ -\frac{1}{a} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & a \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then $\{\mathrm{a}(g_i)\}_{i\in\mathbb{N}}$ diverges since

$$\mathrm{a}(g_{2k-1}) = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & a \end{bmatrix} \qquad \text{but} \qquad \mathrm{a}(g_{2k}) = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}.$$

Some other examples showing that $\{\mathrm{k}(g_i)\}_{i\in\mathbb{N}}$ and $\{\mathrm{a}(g_i)\}_{i\in\mathbb{N}}$ can diverge when $\mathrm{b}(g)$ is not regular for $g \in \mathrm{SL}_n(\mathbb{R})$ can be found in [6, Section 3 and Section 4].

(3) In Theorem 4.7 the element y (depending on g) is not unique; neither is s. However $c_s := (m_s m)^{-1} c(m_s m)$ and $b_s := s^{-1} \cdot b$ as limits are independent of the choice of y or s according to Theorem 4.7.

Example 4.9. We illustrate Theorems 4.5 and 4.7 with G the real symplectic group ([15, p. 129], [17, p. 265]):

$$G := \mathrm{Sp}_n(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) : g^T J_n g = J_n\}, \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Using block multiplication, one finds that the elements of G are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I_n$$

[15, p. 128].

The Iwasawa decomposition of $\mathrm{Sp}_n(\mathbb{R})$ is given by $\mathrm{Sp}_n(\mathbb{R}) = KAN$, where

$$K = \left\{ \begin{pmatrix} C & B \\ -B & C \end{pmatrix} : C + iB \in \mathrm{U}(n) \right\} = \mathrm{O}(2n) \cap \mathrm{Sp}_n(\mathbb{R}),$$
$$A = \{\mathrm{diag}\,(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\},$$
$$N = \left\{ \begin{pmatrix} C & B \\ 0 & (C^{-1})^T \end{pmatrix} : C \text{ unit upper triangular, } CB^T = BC^T \right\}$$

[17, p. 285]. The centralizer M of A in K is the group of the diagonal matrices in K, i.e., the group of matrices of the form $\mathrm{diag}\,(C, C)$, where $C = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$ (independent signs here and below). The normalizer M' of A in K is $W'M$ where W' is generated by

$$\{E_{k,n+k} - E_{n+k,k} + \sum_{i\neq k,n+k} E_{ii} \;:\; k = 1,\dots,n\}$$
$$\cup\,\{\operatorname{diag}(C,C) \;:\; C \text{ is a permutation matrix}\}.$$

Note that $W'M/M \simeq W'/(W' \cap M)$ is isomorphic to the Weyl group. We have $N^- = N^T = \{n^T \,:\, n \in N\}$.

Let $g \in G$ and assume the hypotheses of Theorem 4.5, that is, assume that $ygy^{-1} = cb$ for some $y \in \mathrm{Sp}_n(\mathbb{R})$, $c \in K$ and $b = \operatorname{diag}(b_1,\dots,b_n,b_1^{-1},\dots,b_n^{-1}) \in A$ with $bc = cb$, and that y has Bruhat decomposition $y = n^- m_s man \in N^- m_s MAN$.

Case 1. Suppose that b is regular, i.e., $b \in A_+^\circ$. Choosing A_+, as usual, to be the set of those matrices in A with first n diagonal entries nonincreasing and ≥ 1, we have $b_1 > b_2 > \cdots > b_n > 1$. Since $bc = cb$, it follows that $c = \operatorname{diag}(C,C)$ with $C = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1)$.

According to Theorem 4.7,

(1) $\lim_{i\to\infty} \mathrm{k}(g_i) = c_s \in M$, where $c_s := (m_s m)^{-1} c (m_s m)$ is of the form $\operatorname{diag}(C,C)$ with $C = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1)$,

(2) $\lim_{i\to\infty} \mathrm{a}(g_i) = b_s \in A$, where $b_s := (m_s m)^{-1} b (m_s m) = m_s^{-1} b m_s$ is of the form $\operatorname{diag}(D, D^{-1})$ with D a diagonal matrix having diagonal entries $b_1^{\pm 1}, b_2^{\pm 1}, \dots, b_n^{\pm 1}$ in some order.

The diagonal entries of $\mathrm{n}(g_i)$ are each 1, so it follows that the diagonal entries of the sequence $\{g_i\}$ converge to the eigenvalues of g.

Case 2. Remove the assumption that b is regular. By analogy with Theorem 4.1 (cf. Remark 4.6(2)), assume that the size of E_{θ_i} is γ_i and

$$\begin{aligned} c &= \operatorname{diag}(E_{\theta_1}, \cdots, E_{\theta_m}, E_{-\theta_1}, \cdots, E_{-\theta_m}) \in K, \\ b &= \operatorname{diag}(\lambda_1 I_{\gamma_1}, \dots, \lambda_m I_{\gamma_m}, \lambda_1^{-1} I_{\gamma_1}, \dots, \lambda_m^{-1} I_{\gamma_m}) \in A_+ \end{aligned}$$

(both block diagonals relative to the partition γ). Note that b is not regular if $\theta_i \notin \{0, 1\pi\}$ for some i. Assume further that in the Bruhat decomposition $y = n^- m_s man$ of y the matrix m_s is a block permutation matrix relative to γ. Now n_0^- is block diagonal relative to γ, so $\tilde{c}_s := (n_0^- m_s m)^{-1} c (n_0^- m_s m)$ is block diagonal as well. Thus $AN\tilde{c}_s AN$ consists of matrices of the form

$$\begin{pmatrix} C & B \\ 0 & (C^{-1})^T \end{pmatrix},$$

where C is an upper triangular block matrix relative to γ and $CB^T = BC^T$.

Finally, we point out that since $g \in \mathrm{Sp}_n(\mathbb{R}) \subseteq \mathrm{SL}_{2n}(\mathbb{R})$, we have the Iwasawa decomposition (QR decomposition) $g = k'a'n'$ of g in $\mathrm{SL}_{2n}(\mathbb{R})$,where k' is special orthogonal, a' is positive diagonal, and n' is unit upper triangular. But this is not the Iwasawa decomposition $g = kan$ of g in $\mathrm{Sp}_n(\mathbb{R})$. The QR iteration of g will only yield $\mathrm{k}'(g_i) \in \mathrm{SO}(2n)$ and $\mathrm{n}'(g_i)$ unit upper triangular, but $\mathrm{k}'(g_i) \notin K$ and $\mathrm{n}'(g_i) \notin N$ in general.

5. Cholesky iteration: the regular case

Let Y be a positive definite matrix. The Cholesky decomposition theorem asserts that $Y = R^*R$ where R is an upper triangular matrix with positive diagonal entries. The spectral theorem asserts that there is a unitary matrix U such that UYU^{-1} is diagonal with nonincreasing diagonal entries.

These theorems have counterparts in the Lie group setting. Indeed, by [4, p. 272-273], the map

$$AN \to P, \qquad an \mapsto (an)^*an = n^*a^2n, \tag{18}$$

is a homeomorphism, so each $p \in P$ can be uniquely written as

$$p = n^*a^2n, \qquad a \in A, \quad n \in N.$$

We put

$$\mathrm{r}(p) = an$$

and call $p = \mathrm{r}(p)^*\mathrm{r}(p)$ the Cholesky decomposition of $p \in P$. It follows that P is a subset of the open submanifold N^-MAN of G and that the indicated decomposition is the Bruhat decomposition of p. Moreover, $p = k^{-1}bk$ for some $k \in K$ where $b \in A_+$ is uniquely determined by p, as follows from the decomposition [8, p. 320]

$$\mathfrak{p} = \mathrm{Ad}\,(K)\mathfrak{a}_+.$$

In particular, every $p \in P$ is hyperbolic.

The Cholesky iteration of $p \in P$ is the sequence $\{\tilde{p}_i\}_{i\in\mathbb{N}} \subset P$ given by

$$\begin{aligned} \tilde{p}_1 &:= p, \\ \tilde{p}_i &:= r_{i-1}r_{i-1}^*, \quad i = 2, 3, \ldots, \end{aligned}$$

where $r_i := \mathrm{r}(\tilde{p}_i)$. Since

$$\tilde{p}_i := r_{i-1}\tilde{p}_{i-1}r_{i-1}^{-1}$$

we have

$$\tilde{p}_i := (r_{i-1} \cdots r_1)p(r_{i-1} \cdots r_1)^{-1}, \tag{19}$$

so $\tilde{p}_i$ is conjugate to p for every $i \in \mathbb{N}$.

The following provides a relationship between the Iwasawa iteration $\{p_i\}_{i\in\mathbb{N}}$ of p and the Cholesky iteration $\{\tilde{p}_i\}_{i\in\mathbb{N}}$ of p(cf. [18, p. 546]).

Theorem 5.1. If $p \in P$, then $p_{i+1} = \tilde{p}_{2i+1}$ for $i = 0, 1, 2, \cdots$.

Proof. We claim that

$$p^i = r_1^* \cdots r_i^* (r_i \cdots r_1) = (r_i \cdots r_1)^* (r_i \cdots r_1) \tag{20}$$

for all $p \in P$ and all $i \in \mathbb{N}$. Let $p \in P$. Since $p^1 = \tilde{p}_1 = r_1^* r_1$, the equation holds for $i = 1$. Suppose the equation holds for $i = t - 1$. Applying the induction hypothesis to $\tilde{p}_2$, we get

$$p^t = (r_1^* r_1)^t = r_1^* (r_1 r_1^*)^{t-1} r_1 = r_1^* \tilde{p}_2^{t-1} r_1 = r_1^* r_2^* \cdots r_t^* (r_t \cdots r_2 r_1),$$

so (20) holds for all $i \in \mathbb{N}$.

Let $p \in P$ and $i \in \mathbb{N}$. We have

$$p^{2i} = (p^i)^* p^i = \left[\mathrm{a}(p^i)\mathrm{n}(p^i)\right]^* \mathrm{a}(p^i)\mathrm{n}(p^i), \tag{21}$$

where $p^i = \mathrm{k}(p^i)\mathrm{a}(p^i)\mathrm{n}(p^i)$ is the Iwasawa decomposition of p^i. From (20) and (21) and the uniqueness of the Cholesky decomposition,

$$r_{2i} \cdots r_1 = \mathrm{a}(p^i)\mathrm{n}(p^i).$$

Finally, using (9) we get

$$p_{i+1} = \mathrm{a}(p^i)\,\mathrm{n}(p^i)\,p\,[\mathrm{a}(p^i)\,\mathrm{n}(p^i)]^{-1} = (r_{2i} \cdots r_1)\,p\,(r_{2i} \cdots r_1)^{-1} = \tilde{p}_{2i+1},$$

the last equality from (19). □

Suppose that $p \in P$ is conjugate to a regular element $b \in A_+$. Then $ypy^{-1} = b$ for some $y \in K$. Moreover, by Lemma 3.1, $y'py'^{-1} = ypy^{-1} = b$ for some $y' \in G$ if and only if $y' = m'y \in K$ for some $m' \in M$.

Theorem 5.1 leads to the following asymptotic result about the Iwasawa iteration of p and the Cholesky iteration of p.

Theorem 5.2. Let $p \in P$ and assume that $ypy^{-1} = b$ for some regular $b \in A_+$ and some $y \in K$. Let

$$y = n^- m_s m a n \in N^- m_s M A N$$

be a Bruhat decomposition of y. Then

$$\lim_{i\to\infty} \tilde{p}_i = \lim_{i\to\infty} p_i = s^{-1} \cdot b.$$

Proof. Applying Theorem 4.7 with

$$g = p, \qquad c = 1, \qquad \bar{k} = y^{-1} m_s m, \qquad \text{and} \qquad \bar{a} = \bar{n} = 1,$$

we get

$$\lim_{i\to\infty} \mathrm{k}(p_i) = c_s = 1, \qquad \lim_{i\to\infty} \mathrm{a}(p_i) = b_s = s^{-1}\cdot b,$$

and

$$\lim_{i\to\infty} \mathrm{n}(p_i) = \lim_{i\to\infty} \left(c_s^{i-1}\, \mathrm{n}(p_i)\; c_s^{-i+1}\right) = b_s^{-1}\bar{a}\bar{n}b_s c_s \bar{n}^{-1}\bar{a}^{-1}c_s^{-1} = 1.$$

Therefore, $\lim_{i\to\infty} p_i = s^{-1}\cdot b$.

Theorem 5.1 implies that $\lim_{i\to\infty} \tilde{p}_{2i+1} = s^{-1}\cdot b$. Since (18) is a homeomorphism, the Cholesky iteration function

$$q := r^* r \in P \quad\longmapsto\quad r\in AN \quad\longmapsto\quad \tilde{q}_1 := rr^*$$

is continuous. Therefore, $\lim_{i\to\infty} \tilde{p}_{2i+2}$ converges to the Cholesky iteration of $s^{-1}\cdot b$, which is $s^{-1}\cdot b$. We conclude that $\lim_{i\to\infty} \tilde{p}_i = s^{-1}\cdot b$. □

Let $g\in G$. Recall that $g = \mathrm{k}(g)\mathrm{a}(g)\mathrm{n}(g)$ denotes the Iwasawa decomposition of g. The AN-**sequence** of g is the sequence $\{\tilde{r}_i\}_{i\in\mathbb{N}}$ given by

$$\begin{aligned} \tilde{r}_1 &:= \mathrm{a}(g)\mathrm{n}(g), \\ \tilde{r}_i &:= \mathrm{a}(\tilde{r}_{i-1}^*)\mathrm{n}(\tilde{r}_{i-1}^*), \qquad i = 2,3,\ldots \end{aligned}$$

For example, if $G = \mathrm{SL}_n(\mathbb{R})$ and if $X = QR \in G$, then

$$\begin{aligned} \tilde{r}_1 &:= R, \\ \tilde{r}_i &:= R(\tilde{r}_{i-1}^*), \qquad i = 2,3,\ldots \end{aligned}$$

where $R(\tilde{r}_{i-1}^*)$ is the R-component of $\tilde{r}_{i-1}^*$ in QR decomposition. Note that the singular values of X, but not the eigenvalues, are preserved during this iteration.

The AN-sequence $\{\tilde{r}_i\}_{i\in\mathbb{N}}$ of $g\in G$ is related to the Cholesky iteration $\{\tilde{p}_i\}_{i\in\mathbb{N}}$ of $p := g^*g \in P$ as follows.

Theorem 5.3. Let $\{\tilde{r}_i\}_{i\in\mathbb{N}}$ be the AN-sequence of $g\in G$. Let $\{\tilde{p}_i\}_{i\in\mathbb{N}}$ be the Cholesky iteration of $p := g^*g\in P$. Then $\tilde{p}_i = \tilde{r}_i^*\tilde{r}_i$ for all $i\in\mathbb{N}$.

Proof. We have

$$\tilde{p}_1 = g^*g = [\mathrm{k}(g)\mathrm{a}(g)\mathrm{n}(g)]^*[\mathrm{k}(g)\mathrm{a}(g)\mathrm{n}(g)] = [\mathrm{a}(g)\mathrm{n}(g)]^*[\mathrm{a}(g)\mathrm{n}(g)] = \tilde{r}_1^*\tilde{r}_1,$$

so the equation holds for $i = 1$. Assuming the equation holds for $i = t-1$, we get

$$\begin{aligned} \tilde{p}_t &= \tilde{r}_{t-1}\tilde{r}_{t-1}^* = [\tilde{r}_{t-1}^*]^*\tilde{r}_{t-1}^* \\ &= [\mathrm{k}(\tilde{r}_{t-1}^*)\mathrm{a}(\tilde{r}_{t-1}^*)\mathrm{n}(\tilde{r}_{t-1}^*)]^*[\mathrm{k}(\tilde{r}_{t-1}^*)\mathrm{a}(\tilde{r}_{t-1}^*)\mathrm{n}(\tilde{r}_{t-1}^*)] \\ &= [\mathrm{a}(\tilde{r}_{t-1}^*)\mathrm{n}(\tilde{r}_{t-1}^*)]^*[\mathrm{a}(\tilde{r}_{t-1}^*)\mathrm{n}(\tilde{r}_{t-1}^*)] \\ &= \tilde{r}_t^*\tilde{r}_t. \end{aligned}$$

Therefore, $\tilde{p}_i = \tilde{r}_i^* \tilde{r}_i$ for all $i \in \mathbb{N}$. □

It is well known [8, p. 397] that

$$G = KA_+K.$$

(When $G = \mathrm{SL}_n(\mathbb{R})$, this gives the singular value decomposition of $X \in G$.) We use this decomposition to obtain the asymptotic behavior of the AN-sequence of $g \in G$.

Theorem 5.4. Let $g \in G$ and write $g = xa_+y$ with $x, y \in K$, $a_+ \in A_+$, and $y \in N^- m_s MAN$. Then

$$\lim_{i\to\infty} \tilde{r}_i = s^{-1} \cdot a_+,$$

where $\{\tilde{r}_i\}_{i\in\mathbb{N}}$ is the AN-sequence of g.

Proof. Let $p := g^*g = y^*a_+^2 y = y^{-1}a_+^2 y$. By Theorems 5.3 and 5.2,

$$\lim_{i\to\infty} \tilde{r}_i^* \tilde{r}_i = \lim_{i\to\infty} \tilde{p}_i = s^{-1} \cdot a_+^2 = (s^{-1} \cdot a_+)^2.$$

Since Cholesky decomposition (18) is a homeomorphism, we get

$$\lim_{i\to\infty} \tilde{r}_i = s^{-1} \cdot a_+$$

as desired. □

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